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Asymptotic reductions of two coupled (2 + 1)-dimensional nonlinear Schrödinger equations: application to Bose–Einstein condensates

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Abstract

We consider a system of two coupled (2 + 1)-dimensional nonlinear Schrödinger equations, describing two-component disc-shaped Bose–Einstein condensates. We present three different asymptotic reductions of this system. In particular, we derive the Mel'nikov system, the Yajima–Oikawa system as well as the Davey–Stewartson system (the latter is found as a special case of the Djordjevic–Redekopp system). Conditions for integrability of the reduced systems, their soliton solutions and the asymptotic relevance of such solutions to the original system are also discussed. Numerical results pertaining to the reduction to the Davey–Stewartson system are found to be in good agreement with the analytical predictions.

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1. Introduction

The method of multiscale expansions is a powerful technique that is commonly used in the theory of nonlinear waves, especially in cases characterized by the presence of several different scales; in such cases, this technique usually leads to asymptotic evolution equations more adequate to a given problem [1]. Using this method, it has been shown that several systems integrable by the inverse scattering transform (IST) (see, e.g., [2]) can be reduced to other integrable equations [3]. On the other hand, the multiscale expansion method has proved to be extremely useful in the studies of soliton dynamics in non-integrable systems appearing in various branches of physics. Below we will discuss two particular examples, namely the rapidly growing fields of nonlinear optics [4] and Bose–Einstein condensates (BECs) [5], where the appropriate physical model is the nonlinear Schrödinger (NLS) equation and its variants.

In the context of nonlinear optics, the multiscale expansion method has been used to show that the defocusing NLS equation can be asymptotically reduced to a Korteweg–de Vries (KdV) equation [6], or the Kadomtsev–Petviashvilli (KP) equation [7], in the (1 + 1)- or (2 + 1)-dimensional geometry, respectively. This fact provided for a better insight into the physics of optical solitons as it was possible to study their dynamics in generalized [8] or strongly perturbed [9, 10] NLS models, as well as to analyse their transverse instabilities [11] (see also the reviews [12] and references therein). Asymptotic reductions of coupled NLS equations, which were used to study vector solitons, were reported as well. This way, the Zakharov system and the Mel'nikov system were respectively derived in [13] and [14] to describe the dynamics of a bright–dark or a bright–antidark soliton pair. Other relevant studies, but dealing with solitons in optical media with a quadratic nonlinearity, were reported as well [15, 16].

The progress in the field of nonlinear optics had an impact on studies of coherent nonlinear excitations of matter waves appearing in the context of BECs. In particular, by means of the multiscale expansion method, it was shown that shallow dark matter-wave solitons governed by the Gross–Pitaevskii (GP) equation [5] can effectively be described by means of a KdV [17] or a KP equation [18], in (1 + 1)- or (2 + 1)-dimensions, respectively. These asymptotic reductions were also used to study the oscillations of dark solitons in the BEC regime [19] (see also the recent work in [20]) and the Tonks–Girardeau regime [21]. Recently, the multiscale expansion method was also used in the BEC context to analyse nonlinear excitations, such as one-dimensional vector dark solitons [22] and two-dimensional solitons (dromions) [23]; in the former case the analysis was based on the reduction of two coupled GP equations to two coupled KdV equations, while in the latter case on the reduction of a (2 + 1)-dimensional GP equation to the Davey–Stewartson-I system.

Although there is a significant amount of work that has been done for single-component NLS models in one or higher dimensions, or for vector NLS models in one dimension, much fewer results pertaining to vector NLS models in multidimensions have been reported (see, e.g., [15, 16], as well as [24] where a general criterion for the stability of multi-component solitary waves was proposed). The scope of this paper is to contribute in this direction and present some asymptotic expansions for a system of two coupled, (2 + 1)-dimensional NLS equations appearing in the context of BECs.

In particular, we consider a binary mixture of disc-shaped BECs, consisting of two different spin states of the same isotope, such as, e.g., ^{87}Rb [25] or ^{23}Na [26]. This system is described by the following dimensionless coupled NLS equations (see [27] and the reviews [28]):

$$\mathrm{i}\partial_t u = \left[-\tfrac{1}{2}\Delta + \alpha|u|^2 - \beta|v|^2 + V(x, y)\right]u, \tag{1}$$

$$\mathrm{i}\partial_t v = \left[-\tfrac{1}{2}\Delta - \beta|u|^2 - \gamma|v|^2 + V(x, y)\right]v. \tag{2}$$

In these equations, u and v describe the mean-field wavefunctions of the two-component BEC, under very weak trapping conditions along the x–y plane in comparison with the z-direction [5]. Moreover, $\Delta \equiv \partial_x^2 + \partial_y^2$ is the Laplacian, which, for the disc-shaped BEC under consideration, is taken to be two dimensional [27, 28]. The nonlinearity coefficients α, γ and β in equations (1), (2) describe the intra-species and inter-species interactions respectively and are proportional to the corresponding scattering lengths; note that the positive (negative) scattering length corresponds to repulsive (attractive) atom–atom interaction. Note that equations (1), (2) also appear in the context of optics, where they describe the complex electric field envelopes of two different polarizations or wavelengths [4] (in this case, the variable t is the propagation distance along the optical medium); in this context, the magnitudes and signs of the nonlinearity coefficients are usually fixed (typically, they are all such that all interactions are self-focusing). Nevertheless, in the BEC context, both the magnitude and sign of the nonlinearity coefficients (i.e. the scattering lengths) can be controlled via the Feshbach resonance by an external spatially uniform magnetic [29] or optical [30] field (this also holds for the coupling coefficient β [31]). Finally, the trapping potential $V(x, y)$ in equations (1), (2) is typically harmonic, having, e.g., the form $V(x, y) = (1/2)\Omega^2 r^2$, where $r^2 = x^2 + y^2$ and Ω is a dimensionless parameter. In the case of disc-shaped BECs under consideration, this parameter is defined as the ratio of the confining frequencies in the r- and z-directions and is often small (of order $O(10^{-2})$ [27, 28]).

In what follows, we will consider the homogeneous version of equations (1), (2), i.e. we will neglect the potential terms (see a relevant discussion below). We aim at showing that, in the small-amplitude limit, three different asymptotic reductions of the homogeneous NLS system are possible. In particular, we will obtain the Mel'nikov system, the Yajima–Oikawa system and the Djordjevic–Redekopp system; for the latter, we obtain conditions for its further reduction to the Davey–Stewartson system. The above-mentioned systems appear in the contexts of plasma physics and water waves and have attracted attention in the theory of nonlinear waves [2]. Our results concerning the Mel'nikov and the Davey–Stewartson systems generalize the previous ones, reported in [14] and [13], respectively, that were obtained in $(1 + 1)$-dimensional NLS models. Additionally, the asymptotic reduction of the $(2 + 1)$-dimensional vector NLS model to the Yajima–Oikawa system appears, to the best of our knowledge, for the first time. We will discuss the integrable cases in our reductions, as well as the corresponding soliton solutions, especially in the framework of the original physical problem.

Our analysis will be performed in the absence of the potential terms in equations (1), (2). Although, strictly speaking, such a consideration restricts the validity of our analysis to the homogeneous NLS system, or, in physical terms, to an untrapped binary BEC, the following remarks should be made. First, even in the presence of the potential terms, our results are still approximately valid but only locally, and particularly in a spatial region close to the minimum of the potential ($r \to 0$). Second, the effect of the inhomogeneity due to the presence of the potential terms may still, in principle, be treated analytically, such as, e.g., in the case of [19, 21]. In these studies, the asymptotic reduction of the GP equation (incorporating the trapping potential) led to KdV equations with variable coefficients. Similarly, in our case, we expect that inclusion of the potential terms will lead to inhomogeneous versions of the Mel'nikov, Yajima–Oikawa and Davey–Stewartson systems. In fact, to analyse the effect of external potentials, one should utilize techniques similar to those used in [19, 21] and perform systematic numerical simulations; however, such a detailed study is beyond the scope of the present work.

Finally, as far as the nonlinearity coefficients are concerned, we will assume that $\alpha > 0$, i.e. the u-component is repulsive, and let the signs of γ and β be arbitrary. This way, our analysis is valid in either of the cases $\gamma < 0$ or $\gamma > 0$, corresponding, respectively, to a repulsive or an attractive v-component. In particular, referring to binary BECs, the cases $\alpha > 0$ and $\gamma < 0$ are relevant, e.g., to a mixture of different spin states of ^{87}Rb [25] or ^{23}Na [26]; also, $\alpha > 0$ and $\gamma > 0$ are possible, e.g., in the ^{39}K–^{87}Rb BEC mixture [32], where the rubidium (potassium) features repulsive (attractive) intra-species interaction. Note that in the latter case (and, more generally, for a binary BEC consisting of two different species), the kinetic energy term in equation (2), proportional to the Laplacian, should be multiplied by the ratio of the atomic masses of the two species (see, e.g., [33]); that parameter would not significantly affect our results (apart from some differences in the necessary algebraic conditions involved in the derivation of the reduced systems, the final results would qualitatively be the same). Finally, in all the above cases, the coefficient β describing the inter-species interaction is repulsive, i.e. $\beta < 0$ [25, 26, 31, 32, 34]; in our analysis, however, we will consider the more general case with $\beta \in \Re$.

The paper is organized as follows. In section 2 we first analyse the linear regime of the homogeneous NLS system and then consider the nonlinear regime, for which we present the three different asymptotic reductions. In section 3, we present numerical results to test the validity of our analysis using the example of the asymptotic reduction to the Davey–Stewartson system. Finally, in section 4 we further elaborate the physical relevance of our findings and summarize the conclusions of this work.

2. The asymptotic reductions

We consider the homogeneous version of equations (1), (2), namely,

$$\mathrm{i}u_t + \tfrac{1}{2}\Delta u - \alpha|u|^2 u + \beta|v|^2 u = 0, \tag{3}$$

$$\mathrm{i}v_t + \tfrac{1}{2}\Delta v + \beta|u|^2 v + \gamma|v|^2 v = 0, \tag{4}$$

where the subscripts denote partial derivatives. We introduce the Madelung transformation,

$$u = \sqrt{n}\exp(\mathrm{i}\theta), \tag{5}$$

where n and ϕ denote the normalized density and phase of the u component, respectively. Then, substituting equation (5) into equation (3), we obtain the following set of hydrodynamic equations coupled with equation (4):

$$n_t + \nabla(n\nabla\theta) = 0, \tag{6}$$

$$\theta_t + \frac{1}{2}(\nabla\theta)^2 - \frac{1}{2\sqrt{n}}\Delta\sqrt{n} + \alpha n - \beta|v|^2 = 0, \tag{7}$$

$$\mathrm{i}v_t + \frac{1}{2}\Delta v + \beta n v + \gamma|v|^2 v = 0. \tag{8}$$

A simple solution to equations (6)–(8) is $n = 1, \theta = -\alpha t$ and $v = 0$, namely a continuous-wave (cw) solution of equation (3) and the trivial solution of (4). To describe the linear regime around this solution, we seek for solutions of equations (6)–(8) of the form $n = 1 + \varepsilon r, \theta = -\alpha t + \varepsilon\phi$ and $v = \varepsilon q\exp(\mathrm{i}\beta t)$, where r and ϕ are perturbations and ε is a formal small parameter. This way, equations (6)–(8) are reduced, to $\mathrm{O}(\varepsilon)$, to the following system:

$$r_t + \Delta\phi = 0, \tag{9}$$

$$\phi_t - \tfrac{1}{4}\Delta r + \alpha r = 0, \tag{10}$$

$$\mathrm{i}q_t + \tfrac{1}{2}\Delta q = 0. \tag{11}$$

It is clear that equations (9) and (10) can be further reduced to the following equation:

$$r_{tt} - \alpha\Delta r + \tfrac{1}{4}\Delta^2 r = 0. \tag{12}$$

Assuming that the fundamental excitations of the system are characterized by a frequency ω and a wave number $k \equiv \sqrt{k_x^2 + k_y^2}$, it is clear that equation (12) leads to a phase velocity $c^2 \equiv (\omega/k)^2 = \alpha + k^2/4$. Thus, in the long-wavelength limit ($k \to 0$), small-amplitude waves can propagate along the cw solution with the following 'speed of sound':

$$c^2 = \alpha > 0. \tag{13}$$

Note that the condition $\alpha > 0$ also ensures the modulational stability of the aforementioned cw solution. Additionally, it is readily seen that equation (11) (which is decoupled from equations (9), (10)) admits a plane wave solution of the form $q = \exp[\mathrm{i}(k_x x + k_y y - \omega t)]$, where the frequency ω and the wavenumbers k_x, k_y satisfy the dispersion relation $\omega = \frac{1}{2}\left(k_x^2 + k_y^2\right)$.

The above analysis of the linear regime will be used below for obtaining the various asymptotic reductions of equations (3), (4).

2.1. The Mel'nikov system

To reduce equations, (3), (4) to the Mel'nikov system, we introduce the following stretched variables:

$$X = \varepsilon^{1/2}(x - \sqrt{\alpha}t), \qquad Y = \varepsilon y, \qquad T = \varepsilon^{3/2}t, \tag{14}$$

where ε is a formal small parameter connected to the amplitude of the solutions (see below). We also consider small perturbations around the solution $n = 1, \theta = -\alpha t$ and $v = 0$, which are functions of the above-stretched variables. We thus seek solutions of equations (6)–(8) of the following form:

$$n = 1 + \varepsilon r(X, Y, T), \tag{15}$$

$$\theta = -\alpha t + \varepsilon^{1/2}\phi(X, Y, T), \tag{16}$$

$$v = \varepsilon q(X, Y, T)\exp(\mathrm{i}\beta t)\exp[\mathrm{i}(k_x x + k_y y - \omega t)], \tag{17}$$

where $\omega = \frac{1}{2}\left(k_x^2 + k_y^2\right)$ as earlier. This way, the following system is obtained for the unknown perturbations r, ϕ and q:

$$-\sqrt{\alpha}r_X + \phi_{XX} + \varepsilon[r_T + (r\phi_X)_X + \phi_{YY}] + O(\varepsilon^2) = 0, \tag{18}$$

$$-\sqrt{\alpha}\phi_X + \alpha r + \varepsilon\left[\phi_T + \frac{1}{\alpha}(\phi_X)^2 - \frac{1}{4}r_{XX} - \beta|q|^2\right] + O(\varepsilon^2) = 0, \tag{19}$$

$$\mathrm{i}k_y q_Y + \frac{1}{2}q_{XX} + \beta r q + \mathrm{i}\varepsilon^{1/2} q r + \varepsilon\left[\frac{1}{2}q_{YY} + \gamma\,|q|^2\,q\right] = 0, \tag{20}$$

where $k_x = \sqrt{\alpha}$. To obtain self-consistent equations, we set $\phi_X = \sqrt{\alpha}r + \varepsilon\Phi_X$, which leads to the result

$$\Phi_{XX} + r_T + \sqrt{\alpha}(r^2)_X + \sqrt{\alpha}\partial_X^{-1}r_{YY} + O(\varepsilon) = 0, \tag{21}$$

$$-\sqrt{\alpha}\Phi_X + \sqrt{\alpha}\partial_X^{-1}r_T + \alpha r^2 - \tfrac{1}{4}r_{XX} - \beta|q|^2 + O(\varepsilon) = 0, \tag{22}$$

$$\mathrm{i}k_y q_Y + \tfrac{1}{2}q_{XX} + \beta r q + O(\varepsilon^{1/2}) = 0. \tag{23}$$

Finally, eliminating the function Φ_X and rescaling the variables as $r = (-3/\sqrt{\alpha})R, q = \sqrt{6/|\beta|}Q, \xi = \sqrt{8\sqrt{\alpha}}X, \eta = (-4\sqrt{\alpha}/k_y)Y$ and $\tau = -\sqrt{8\sqrt{\alpha}}T$, we end up to the following dimensionless system:

$$(R_\tau + 6RR_\xi + R_{\xi\xi\xi})_\xi - 3R_{\eta\eta} + \sigma|Q|^2_{\xi\xi} = 0, \tag{24}$$

$$\mathrm{i}Q_\eta = Q_{\xi\xi} + \chi R Q, \tag{25}$$

where $\chi = -3\beta/4\alpha$ and $\sigma = -\mathrm{sign}(\beta)$. The system of equations (24) and (25) is composed of a KP equation with a self-consistent source satisfying a Schrödinger equation and is known as the (2 + 1)-dimensional Mel'nikov system [35]. It has also been suggested in (1 + 1)-dimensions, where it has the form of a KdV equation coupled with a stationary Schrödinger equation [36]; in this case, the Mel'nikov system is an asymptotic reduction of a system describing coupled nonlinear electron-plasma and ion-acoustic waves [37].

The Mel'nikov system is integrable only for $\chi = 1$, i.e. for $\beta = -(4/3)\alpha < 0$. It should be noted that although this is a special case in our analysis, it could be realized in a real physical system, namely in a rubidium–potassium BEC mixture, upon properly tuning the magnitude of the respective atom–atom interactions through the Feshbach resonance mechanism. This would be particularly interesting as it would permit us to tunably access an integrable limit of the corresponding dynamical equations (in the regime, of course, where the above reductions are valid). This, in turn, allows for soliton solutions which are explicitly available [35]. In this setting, such waves are of the form of a vector dark–bright soliton, in the u- and v-components, respectively. Finally, it should be mentioned that the present result generalizes that presented in [14], referring to the (1 + 1)-dimensional case.

2.2. The Yajima–Oikawa system

We now consider a different asymptotic reduction of equations (3), (4), namely to the (2 + 1)-dimensional Yajima–Oikawa system. Following the procedure of the previous subsection, we first introduce the following stretched variables:

$$X = \varepsilon^{1/2}(x - \sqrt{\alpha}t), \qquad Y = \varepsilon^{3/4}y, \qquad T = \varepsilon t, \tag{26}$$

and then look for solutions of equations (6)–(8) of the following form:

$$n = 1 + \varepsilon r(X, Y, T), \tag{27}$$

$$\theta = -\alpha t + \varepsilon^{1/2}\phi(X, Y, T), \tag{28}$$

$$v = \varepsilon^{3/4}q(X, Y, T)\exp(\mathrm{i}\beta t)\exp[\mathrm{i}(k_x x + k_y y - \omega t)]. \tag{29}$$

Performing similar calculations as in the first reduction, we obtain the following system for the unknown functions r, ϕ and q:

$$-\sqrt{\alpha}r_X + \phi_{XX} + \varepsilon^{1/2}(r_T + \phi_{YY}) + \varepsilon(r\phi_X)_X + O(\varepsilon^{3/2}) = 0, \tag{30}$$

$$-\sqrt{\alpha}\phi_X + \alpha r + \varepsilon^{1/2}(\phi_T - \beta|q|^2) + \varepsilon\left[(\phi_X)^2 - \tfrac{1}{4}r_{XX}\right] + O(\varepsilon^{3/2}) = 0, \tag{31}$$

$$\mathrm{i}q_T + \tfrac{1}{2}q_{XX} + \beta rq + \varepsilon^{1/2}\left(\tfrac{1}{2}q_{YY} + \gamma|q|^2q\right) = 0, \tag{32}$$

where again we have $k_x = \sqrt{\alpha}, k_y = 0$ and $\omega = \frac{1}{2}\left(k_x^2 + k_y^2\right)$. To obtain self-consistent equations, we set $\phi_X = \sqrt{\alpha}r + \varepsilon^{1/2}\Phi_X$ and eliminate Φ_X from the leading order. Then, introducing the scale transformations $\xi = \sqrt{8\sqrt{\alpha}}X, \eta = (8^3\sqrt{\alpha})^{1/4}Y, \tau = 4\sqrt{\alpha}T, r = -(4\sqrt{\alpha}/\beta)R$ and $q = [(2\sqrt{\alpha}/\beta)^4\sqrt{8\sqrt{\alpha}}]Q$, we finally obtain the following dimensionless system:

$$\mathrm{i}Q_\tau + Q_{\xi\xi} - RQ = 0, \tag{33}$$

$$\left(R_\tau + |Q|_\xi^2\right)_\xi + R_{\eta\eta} = 0. \tag{34}$$

Equations (33), (34) constitute the so-called Yajima–Oikawa system [38] in (2 + 1)-dimensions. This is a long–short wave interaction system, which has originally been suggested to describe Langmuir waves coupled with ion-acoustic waves in plasmas. The Yajima–Oikawa system is integrable in the case when $R_\eta = 0$. Soliton solutions can be found in this case [38], which, in our problem, and similarly to the case of the Mel'nikov system, have the form of a vector dark–bright soliton, in the u- and v-components, respectively.

2.3. The Davey–Stewartson system

Finally, using a different scaling, a reduction of the system (3), (4) to the so-called Davey–Stewartson system can be obtained as follows. First, we introduce the stretched variables

$$X = \varepsilon^{1/2}(x - ct), \qquad Y = \varepsilon^{1/2}y, \qquad T = \varepsilon t, \tag{35}$$

where the velocity c is an arbitrary parameter. Then, we consider solutions of equations (6)–(8) having the following form:

$$n = 1 + \varepsilon r(X, Y, T), \tag{36}$$

$$\theta = -\alpha t + \varepsilon^{1/2}\phi(X, Y, T), \tag{37}$$

$$v = \varepsilon^{1/2}q(X, Y, T)\exp(\mathrm{i}\beta t)\exp[\mathrm{i}(k_x x + k_y y - \omega t)]. \tag{38}$$

This way, the following system is obtained:

$$-cr_X + \Delta\phi + \varepsilon^{1/2}r_T + \varepsilon\nabla(r\nabla\phi) = 0, \tag{39}$$

$$-c\phi_X + \alpha r - \beta|q|^2 + \varepsilon^{1/2}\phi_T + \varepsilon\left[(\nabla\phi)^2 - \tfrac{1}{4}\Delta r\right] + O(\varepsilon^{3/2}) = 0, \tag{40}$$

$$\mathrm{i}q_T + \tfrac{1}{2}\Delta q + \beta rq + \gamma|q|^2q = 0. \tag{41}$$

To the leading order, we eliminate r and find relations connecting the parameters involved: $k_x = c, k_y = 0$ and $\omega = \frac{1}{2}\left(k_x^2 + k_y^2\right)$; we also find the following equation connecting the unknown functions ϕ and q:

$$r = \frac{c}{\alpha}\phi_X + \frac{\beta}{\alpha}|q|^2 + O(\varepsilon^{1/2}). \tag{42}$$

Then, introducing the function $V = \phi_X$, after straightforward manipulations, we obtain the following system of equations:

$$\mathrm{i}q_T + \frac{1}{2}\Delta q + \frac{\beta c}{\alpha}Vq + \left(\gamma + \frac{\beta^2}{\alpha}\right)|q|^2q = 0, \tag{43}$$

$$\left(1 - \frac{c^2}{\alpha}\right)V_{XX} + V_{YY} = \frac{c\beta}{\alpha}|q|^2_{XX}. \tag{44}$$

Equations (43), (44) are in the form of the so-called Djordjevic–Redekopp (DR) system, which, in the theory of water waves, describes capillary-gravity waves [39]. This result, i.e. the small-amplitude limit reduction of equations (1), (2) to the DR system, resembles the result obtained in [13], where a vector (1 + 1)-dimensional NLS equation was reduced to the generalized Zakharov equations.

Although the DR system is known to be in general nonintegrable, there exist conditions for integrability (which were found by reducing the DR system to the integrable Davey–Stewartson system by properly selected scale transformations and imposing satisfaction of additional algebraic conditions). Following this procedure in the particular case of equations (43), (44), we introduce the transformations $\xi = X, \eta = Y, \tau = T$, $V = -(\sqrt{2\alpha}/2\beta)R$ and $q = (\sqrt{\alpha}/|\beta|)Q$. Additionally, under the assumption that $c = \sqrt{2\alpha}$ (recall that the parameter c is arbitrary), the DR system (43), (44) is transformed to the form,

$$R_{\xi\xi} - R\eta\eta - 2|Q|^2_{\xi\xi} = 0, \tag{45}$$

$$\mathrm{i}Q_\tau + \tfrac{1}{2}(Q_{\xi\xi} + Q_{\eta\eta}) - RQ + \chi|Q|^2Q = 0, \tag{46}$$

where $\chi = 1+\alpha\gamma/\beta^2$. In the case $\chi = 1$, or for $\gamma = 0$ (since $\alpha \neq 0$), equations (45), (46) are known as the Davey–Stewartson (DS) system, which has originally been derived in the theory of water waves [40]. In fact, the DS system is a shallow-water limit of the Benney–Roskes equations [41], where Q is the amplitude of a surface wave packet and R characterizes the mean flow generated by this surface wave. Note that in addition to fluid dynamics, the DS system has also been obtained in different contexts, such as nonlinear optics [15, 16], BECs [23], plasma physics [42], magnetics [43], lattice dynamics [44], and so on.

The particular elliptic-hyperbolic version of the DS system (45), (46) is known as the DS-I system, which is completely integrable by the inverse scattering transform [2]. In this case, and in the physical context at hand, the most interesting soliton solutions are the localized, exponentially decaying two-dimensional solitons driven by nontrivial boundary conditions, which have been derived for the DS-I equations [45]. These solitons are known as 'dromions' because they travel on the tracks described by the mean flow (note that in Greek 'dromos' means 'track'). In our case, i.e. in the framework of equations (3), (4), such a solution corresponds to a bright 2D soliton (the dromion) in the v-component, driven by two intersecting dark-soliton stripes in the u-component.

As mentioned above, formally such a solution is possible for $\gamma = 0$, i.e. physically speaking, when there is no atom–atom interaction in the v-component. Nevertheless, this may also happen for $\gamma = O(\varepsilon^j)$, with $j \geqslant 1/2$, a choice that transfers the corresponding term $(\alpha\gamma/\beta^2)|Q|^2Q$ to the next order of approximation. In physical terms, such a situation (with weak inter-atomic interactions in the one BEC species) may, once again, be realized by means of the Feshbach resonance mechanism, i.e. upon utilizing proper external magnetic fields to 'drive' the scattering length of the particular species sufficiently close to zero.

Finally, let us analyse some additional possibilities concerning the choice of the arbitrary parameter c. First, if $c = \sqrt{\alpha}$ (the choice made in previous subsections), the DR system (43), (44) reads

$$\mathrm{i}q_T + \frac{1}{2}\Delta q + \frac{\beta}{\sqrt{\alpha}}Vq + \left(\gamma + \frac{\beta^2}{\alpha}\right)|q|^2q = 0 \tag{47}$$

$$\frac{\beta}{\sqrt{\alpha}}|q|^2_{XX} = V_{YY}. \tag{48}$$

This choice is always secular in Y if $|q|^2$ depends only on X. Another choice is $c = 0$, for which we have $V = 0$ and the system is reduced to the $(2+1)$-dimensional NLS equation:

$$\mathrm{i}q_T + \frac{1}{2}\Delta q + \left(\gamma + \frac{\beta^2}{\alpha}\right)|q|^2q = 0. \tag{49}$$

Generally speaking, $c = k_x$ where k_x is the driving parameter.

3. Numerical results

It is clear that the three different asymptotic reductions lead to different types of approximate soliton solutions of the original NLS system of equations (3), (4). The study of the dynamics of these solutions in the framework of the original system is an interesting issue by its own right, and may also be quite relevant to the BEC experiments. In particular, one can use solitonic solutions from the proposed reduced systems to initialize the dynamics of equations (3), (4). As an example, we will consider hereafter the reduction to the DS-I system, which (as mentioned above) gives rise to the exponentially decaying 2D solitons known as 'dromions'. The general dromion solution of equations (45), (46) can be found in [45] (see also [2]). Here, we assume the special case of a dromion solution for the Q-component, expressed in the form

$$Q(\tilde{X},\tilde{Y},T)=\frac{Q_1(\tilde{X},\tilde{Y},T)}{Q_2(\tilde{X},\tilde{Y},T)}, \tag{50}$$

where $\tilde{X}\equiv X+Y$, $\tilde{Y}\equiv X-Y$ and the functions Q_1, Q_2 are given by

$$Q_1=4\mathrm{i}\exp[-(1+\mathrm{i})(\tilde{X}+\tilde{Y})-4T], \tag{51}$$

$$Q_2=1+(1+\exp(-2\tilde{X}-4T))(1+\exp(-2\tilde{Y}-4T)). \tag{52}$$

In the R-component, the boundary conditions associated with the above dromion are

$$R_1(\tilde{X},-\infty,T)=-2\,\mathrm{sech}^2(\tilde{X}+2T), \tag{53}$$

$$R_2(-\infty,\tilde{Y},T)=-2\,\mathrm{sech}^2(\tilde{Y}+2T). \tag{54}$$

With regard to these boundary conditions, equation (45) can be formally integrated to express the mean flow (in the above-mentioned hydrodynamic interpretation) in terms of the local intensity $|Q|^2$ of the surface wave:

$$R(\tilde{X},\tilde{Y},T)=-|Q|^2+R_1+R_2-\frac{1}{2}\left(\int_{-\infty}^{\tilde{X}}(|Q|^2)_{\tilde{Y}}\,\mathrm{d}\tilde{X}'+\int_{-\infty}^{\tilde{Y}}(|Q|^2)_{\tilde{X}}\,\mathrm{d}\tilde{Y}'\right). \tag{55}$$

Now, we may use these expressions, together with the definitions of the fields in equations (36)–(38) and the stretched variables in equation (35), to initialize the approximate soliton solutions of equations (3), (4). Then, we can readily find the relevant initial conditions to numerically integrate the original NLS system and study the dynamics of the solutions.

Following the above procedure, we have then used the split-step Fourier method to numerically integrate equations (3), (4) with the following initial conditions:

$$u(0)=\sqrt{\left[1-\frac{2\varepsilon}{\beta}(|Q_0(\tilde{x},\tilde{y})|^2+\mathrm{sech}^2(\varepsilon^{1/2}\tilde{x})+\mathrm{sech}^2(\varepsilon^{1/2}\tilde{y}))\right]}u_{\mathrm{b}}, \tag{56}$$

$$v(0)=\frac{\sqrt{\varepsilon a}}{\beta}Q_0(\tilde{x},\tilde{y})\exp(\mathrm{i}\sqrt{2a}x), \tag{57}$$

where $\tilde{x}\equiv x+y$, $\tilde{y}\equiv x-y$, and

$$Q_0(\tilde{x},\tilde{y})=\frac{4\mathrm{i}\exp(-\varepsilon^{1/2}(1+\mathrm{i})(\tilde{x}+\tilde{y}))}{(1+\exp(-2\varepsilon^{1/2}\tilde{x}))(1+\exp(-2\varepsilon^{1/2}\tilde{y}))+1}. \tag{58}$$

Note that in equation (56), the background field u_{b}, which would be $u_{\mathrm{b}}\equiv 1$ in the infinite system, in the simulations was taken to be a very broad super-Gaussian of the form $u_{\mathrm{b}}=\exp[-((x^2+y^2)/R^2)^8]$, with $R=200$. This way, we emulated the fact that the background is of finite extent in a real BEC experiment. The condensate is always confined to an external potential (which, here, is assumed to be a box-like one) and, as a result, it always has a finite size determined, e.g. by the Thomas–Fermi approximation [5].

The initial conditions (56), (57) correspond to the dromion in the v-component, and two dark stripes (corresponding to the wave forms R_1 and R_2 in equations (53), (54)) on top of a finite-extent background in the u-component; also, at the point of intersection of these stripes, there exists an additional hump arising from the term $\sim|Q|^2$ in equation (55). For simplicity of computation, we have only used the first three terms of equation (55). The resulting

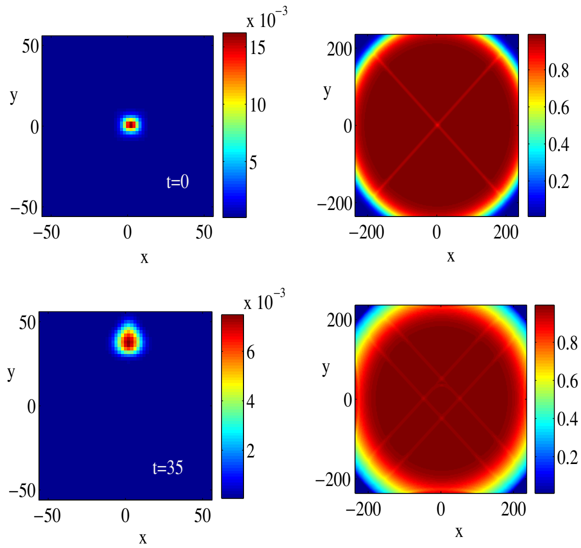

Figure 1. Top panels: contour plots of the density $|v|^2$ of the dromion (left panel) and the density $|u|^2$ of the configuration with the intersecting dark-soliton stripes in the u-component (right panel) at $t = 0$. Parameters in the NLS model are $\alpha = 1$, $\beta = -1$, $\gamma = -0.1$, while $\varepsilon = 0.1$. Bottom panels: same as in top panels but at $t = 35$, obtained by direct numerical integration of the NLS equations. Note that the dromion has been slightly dispersed and has lost ≈45% of its initial power (left panel). On the other hand, each of the dark stripes in the u-component has split into two (right panel). The stripes generated by the splitting travel along the $+y$-direction and drag the dromion's component in the v-field.

(This figure is in colour only in the electronic version)

configuration in the u-component may be identified as a superposition of two intersecting dark solitons and a dark localized hump located at the intersection point. The initial configurations of both the u- and v-fields are shown in the top panels of figure 1; here, the parameters in the coupled NLS equations are $\alpha = 1$, $\beta = -1$ and $\gamma = -0.1$, while the small parameter was $\varepsilon = 0.1$ (for the choice of γ, recall that this parameter should be small so that the reduction to the DS-I system is valid). In physical terms, such a choice may correspond to a spin state mixture of the ^{87}Rb condensate (described by the mean-field wavefunctions u and v), with the scattering length of one state (v-component) set to a relatively small value utilizing the Feshbach resonance control via external magnetic fields. Note that the results of the simulations (see below) were similar, both qualitatively and quantitatively, in the case of $\gamma = +0.1$, a choice corresponding, e.g. to a ^{39}K–^{87}Rb BEC mixture.

The simulations have shown that, strictly speaking, the dromion is unstable, persisting up to $t \approx 100$, while slowly dispersing and decaying. In particular, as shown in the bottom left panel of figure 1, the dromion loses ≈45% of its initial peak power at $t = 35$. Note that corresponding to equations (3), (4), the dimensional time unit is of the order of a millisecond [28]; this result indicates that the dromion may be observed in a real BEC experiment. On the other hand, the evolution of the u-component (i.e. of the two dark stripes carved in the finite-extent background which guide the dromion) is also an interesting issue deserving consideration. To understand this evolution, we note the following. If an additional, properly adjusted, phase was assumed under the initial condition (56) (see the expressions for u and θ in equations (5), (36) and (37)), then, asymptotically as $x, y \to \pm\infty$, these dark stripes could be made true dark solitons. Indeed, one can determine the phase θ via ϕ by integrating equation (42) with respect to X; this way, the integral of R (and, in particular, the integrals of R_1 and R_2 in equations (53) and (54)) will produce terms $\sim \tanh(x+y)$ and $\tanh(x-y)$ in θ, which display characteristic phase jumps of dark solitons (see [9]). However, since such a phase field was not introduced in the simulations (i.e. the actual phase has even instead of odd parity), each of the two stripes splits into two counter-propagating ones, as seen in the bottom right panel of figure 1. In the new configuration generated by the splitting, each dark stripe features a *correct* phase distribution, with the odd symmetry (not shown in the figure), i.e. it is a genuine shallow dark-soliton stripe. Note that the secondary dark solitons (along with the above-mentioned dark hump trapped at their intersection point) travelling along the $+y$-direction steer the dromion's v-component. It should also be mentioned that, due to the splitting process, the dark stripes evolve having half the appropriate amplitude. Numerical simulations that have been performed (not shown here) suggest that if the initial amplitude of the dark stripes is multiplied by a factor of 2, then the dromion decays much slower, losing just $\approx$20% of its peak power at $t = 35$ (as opposed to $\approx$45% shown in figure 1). This means that, in real time units, the lifetime of the dromion is of the order of 100 ms and thus it has indeed a good chance to be observed experimentally.

4. Conclusions and outlook

In this work, we have found several asymptotic reductions of two nonlinearly coupled $(2+1)$-dimensional nonlinear Schrödinger equations. In particular, in the small-amplitude limit, and for different choices of stretched coordinates, we have derived the Mel'nikov system, the Yajima–Oikawa system and the Davey–Stewartson system. The latter was found as a special case of the Djordjevic–Redekopp system. Conditions for integrability of the reduced systems, as well as their soliton solutions have also been discussed.

The results can be applied in the context of BECs in which the considered NLS system describes a two-component disc-shaped condensate. Although our analysis was performed for an untrapped BEC, the results are still approximately valid in the case of a trapped condensate, but locally, i.e. in a spatial region close to the trap's minimum. The results are valid in both cases of repulsive–repulsive and repulsive–attractive species, such as the spin state mixture of ^{87}Rb or ^{23}Na, and the ^{39}K–^{87}Rb BEC mixture, respectively (in both cases, the inter-species interaction is repulsive).

The three reduced systems were derived upon considering different asymptotic scales in space and time, as well as different asymptotic expansions of the unknown fields. Apparently, these scalings characterize (and are characterized by) the exact soliton solutions of each of the reduced models. These solutions can subsequently be used to construct approximate soliton solutions of the original NLS equations. It is, therefore, interesting to consider the differences between the asymptotic regimes and discuss their physical relevance.

In this respect, we first note that none of these scalings is more physically relevant than the other. They are all equally relevant but *in different settings*. As general asymptotic reduction theory dictates, the relevant scalings are appropriate for corresponding temporal and spatial scales, e.g. when in the Mel'nikov system we scale $T = \varepsilon^{3/2}t$, the scaling is relevant for times of $O(\varepsilon^{-3/2})$, when in the Yajima–Oikawa and DS systems we scale time $T = \varepsilon t$, the scaling is valid for times of $O(\varepsilon^{-1})$, and similar considerations are applicable for the respective spatial scales of the reductions.

Then, following the above arguments, a natural question would be: Since ε is nothing but a formal parameter, what *practically* determines these temporal and spatial scales defined above, for which the corresponding reductions are physically relevant? The answer hinges on the scaling of the fields themselves and the number of atoms in each BEC component. In particular, if we initialize a dark–bright soliton of the Mel'nikov system, or of the Yajima–Oikawa system, then the corresponding number of atoms in the bright component ($N_2 = \varepsilon \int |q|^2 \, \mathrm{d}x \, \mathrm{d}y$ in the first case and $N_2 = \varepsilon^{3/2} \int |q|^2 \, \mathrm{d}x \, \mathrm{d}y$ in the second) will determine the relevant scaling parameter ε, and as such the spatial and temporal scales over which we should expect to see such a structure persist. Note, by the way, that if two different scalings give the same type of structure (as the two scalings above both yield a dark–bright solitary wave), the fact that will determine which one of the two is relevant is the profile/number of atoms in the first component (in comparison with those of the second component). In fact, in the Mel'nikov case, the size of the notch of atoms in the centre of the condensate scales similarly to the size of the density of atoms in the second component, while in the Yajima–Oikawa system that is no longer the case. Thus, for a given system (i.e. fixing the magnitudes and signs of α, β and γ), and for a given initial profile/number of atoms in the two components (i.e. fixing the size and scaling of ε in them), we can, based on the above considerations, extract the type of reduction that will be relevant and the spatial and temporal scales over which it will be expected to persist.

Turning back to the 'outcome' of the reductions, let us discuss now the dynamics of the soliton solutions of the reduced systems in the framework of the original NLS system. We have performed numerical simulations in the case of the Davey–Stewartson system and our analytical predictions have been found to be in a fairly good agreement with the numerical results. In particular, we have found that the predicted dromion solution persists up to times relevant to experiments. The dromion loses half of its initial power after propagating for a physical time of order of 100 ms.

It would also be interesting to use solitonic solutions of the Mel'nikov and Yajima–Oikawa systems to initialize the dynamics of the coupled nonlinear Schrödinger system. Such a study could reveal other interesting phenomena due to dimensionality, such as the transverse modulational instability of soliton stripes occurring in a single component $(2+1)$-dimensional NLS equation, leading to vortex formation [7, 11]. In particular, the soliton solutions of the $(1 + 1)$-dimensional versions of the Mel'nikov and Yajima–Oikawa systems, which satisfy their $(2 + 1)$-dimensional counterparts, are expected to be subject to this instability when embedded in the 2D space. Such an investigation would be particularly relevant also in the presence of external trapping potentials, i.e. in the framework of equations (1), (2). Then, if instabilities due to dimensionality manifested themselves, it would be interesting to observe the dynamics of the resulting nonlinear structures, such as vortices (see, e.g. the reviews [28]). However, such a detailed investigation is beyond the scope of this work.

Finally, another interesting direction, also relevant to BECs, is the analytical treatment of equations (1), (2), incorporating the potential terms. Such a study is in principle possible (see [19, 21]) and is expected to lead to inhomogeneous versions of the reduced systems. This way, novel inhomogeneous evolution equations of physical significance could be derived.

Relevant studies, along the lines suggested above, are currently in progress and the results will be reported elsewhere.

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